'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2021

GE2-P1-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains MATHGE1, MATHGE2, MATHGE3, MATHGE4 and MATHGE5. Candidates are required to answer any *one* from the *five* MATHGE courses and they should mention it clearly on the Answer Book.

MATHGE1

CALCULUS, GEOMETRY AND DE

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Prove that the area included between the Folium of Descartes $x^3+y^3=3axy$ and its asymptotes $x+y+a=0$ is $3/2\,a^2$. 3

(b) Discuss the characteristics of the curve $y^2(x^2-9)=x^4$ and then sketch or trace it. 3

(c) Discuss the asymptotes of the curve $y=\frac{3x}{2}\log\left(e-\frac{1}{3x}\right)$. 3

(d) Find the area bounded by the curve $y=\log x$, x-axis and the line $x=10$. 3

(e) Show that origin is the point of inflexion of the curve $a^2y^2=a^2x^2-x^4$. 3

(f) Show that the line $\frac{x+2}{2}=\frac{y}{3}=\frac{z-1}{-2}$ is a generator of the cone $\frac{x^2}{4}-\frac{y^2}{9}=z$. 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) Find the asymptotes of the curve $(x^2-y^2)-8(x^2+y^2)+8x-16=0$. 6

(b) Find the area included between the curve $xy^2=4a^2(2a-x), a>0$ and its asymptotes. 6

(c) A figure bounded by $x^{2/3}+y^{2/3}=a^{2/3}$ is revolved about x-axis. Find the volume of the solid of revolution. 6

(d) Show that for the conic $\frac{l}{r}=1+e\cos\theta$, the equation to the directrix corresponding to the focus other than pole is $\frac{l}{r}=\frac{-(1-e^2)\,e\cos\theta}{(1+e^2)}$. 6

(e) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$. Find the equation of other two generators. 6

(f) If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, $|x|<1$, then show that $(1-x^2)y_{n+2}-(2n+3)xy_{n+1}-(n+1)^2y_n=0$. 6

GROUP-C

Answer any *two* questions from the following 12×2= 24

3. (a) Find the envelope of the curve $x^2\cos\theta+y^2\sin\theta=a^2$, where θ is a parameter. 6

(b) A sphere of radius *r* passes through origin and meets the coordinate axes at *A*, *B*, *C*. Prove that the centroid of triangle *ABC* lies on the sphere $9(x^2+y^2+z^2)=4r^2$. 6

4. (a) If ρ, ρ' be the radii of curvature at the ends of two conjugate diameters of an ellipse, prove that $(\rho^{2/3}+\rho'^{2/3})(ab)^{2/3}=(a^2+b^2)$. 6

(b) Solve $(y^4+2y)dx+(xy^3+2y^4-4x)dy=0$ by evaluating Integrating factor. 6

5. (a) Find *a* and *b* in order that $\lim\limits_{x\to 0}\frac{a\sin 2x-b\sin x}{x^3}=1$. 6

(b) Find the angle through which the axes must be turned so that the equation $lx-my+n=0$ $(m\neq 0)$ may be reduced to the form $ay+b=0$. 6

6. (a) Reduce the equation $xyp^2-p(x^2+y^2-1)+xy=0$ to Clairaut's form by the substitutions $x^2=u$, $y^2=v$. Hence show that the equation represents a family of conics touching four sides of a square. 6

(b) Show that the envelope of the circles whose centre lie on the rectangular hyperbola $xy=c^2$ and which passes through its centre is $(x^2+y^2)^2=16c^2xy$. 6

MATHGE2

ALGEBRA

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Find the nature of the roots of the equation $x^6+x^4+x^2+2x+5=0$, by using Descartes rule of signs. 3

(b) Prove that $3.4^{n+1}\equiv 3 \pmod 9$ for all positive integer *n*. 3

(c) If $A=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$, find the rank of the matrix $A+A^2$. 3

(d) If P is an $n\times n$ real orthogonal matrix with $\det P=-1$. Prove that $P+I_n$ is a singular matrix. 3

(e) For what real values of k, does the set $S=\{(k,1,k),(0,k,1),(1,1,1)\}$ form a basis of $\mathbb{R}^3$? 3

(f) Prove that $\log(3+4i)=\log 5+\left(2n\pi+\tan^{-1}\frac{4}{3}\right)i$. 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) If α,β,γ are the roots of the equation $2x^3+3x^2-x-1=0$, find the equation whose roots are $\frac{\beta+\gamma}{\alpha},\frac{\gamma+\alpha}{\beta},\frac{\alpha+\beta}{\gamma}$. 6

(b) Find the roots of the equation $x^{24}-1=0$. Deduce the values of $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$. 6

(c) If $S_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$, prove that 3+3

(i) $S_n>\frac{2n}{n+1}$, if $n>1$

(ii) $n+S_n>n(n+1)^{1/n}$, if $n>1$.

(d) Prove that the linear mapping $T:\mathbb{R}^3\to\mathbb{R}^3$ that maps the basis vectors (1, 2, 2), (2, 1, 2), (2, 2, 1) of $\mathbb{R}^3$ to the vectors (0, 1, 1), (1, 0, 1), (1, 1, 0) respectively is one-one and onto. 6

(e) Solve the system of linear congruences $x\equiv 2 \pmod 3$, $x\equiv 3 \pmod 5$, $x\equiv 4 \pmod 7$. 6

(f) Let M be a 3×3 real matrix with the eigen values 2, 3, 1 and corresponding eigen vectors (1, 2, 1), (0, 1, 1), (1, 1, 1) respectively. Determine the matrix M. 6

GROUP-C

Answer any *two* questions from the following 12×2= 24

3. (a) If $\tan\log(x+iy)=\alpha+i\beta$, where $\alpha^2+\beta^2\neq 1$, then prove that 6

$$\tan\log(x^2+y^2)=\frac{2\alpha}{1-\alpha^2-\beta^2}$$

(b) Using Euclidean algorithm, find integers u and v such that $1269u+297v=135$. 6

4. (a) If $\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma$, then prove that 4

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\frac{3}{2}$$

(b) Let x, y, z be positive real numbers and $x+y+z=1$. Show that 4

$$8xyz\le(1-x)(1-y)(1-z)\le\frac{8}{27}$$

(c) Solve the linear congruence $15x\equiv 9 \pmod{18}$. 4

5. (a) Find the eigen values and the corresponding eigen vectors of the matrix 6

$$\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & 2 & 0 \end{pmatrix}.$$

Further, find the algebraic and the geometric multiplicities of the eigen values.

(b) A linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = (x-y, x+2y, y+3z)$, $(x, y, z) \in \mathbb{R}^3$. Show that T is non-singular and determine T^{-1}. 6

6. (a) Solve by Cardan's method the equation: $x^3 - 18x - 35 = 0$. 6

(b) Determine the conditions for which the system of equation has 2+2+2

(i) unique solution,

(ii) no solution

(iii) infinitely many solutions

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

MATHGE3

DIFFERENTIAL EQUATION AND VECTOR CALCULUS

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Show that $\frac{1}{D-m} f(x) = e^{mx} \int e^{-mx} f(x)dx$.

(b) To solve a linear homogeneous ordinary differential equation with constant coefficients why do you assume e^{mx} (m is a constant) as a trial solution?

(c) Show that $x = 0$ is an ordinary point of $(x^2-1)y'' + xy' - y = 0$, but $x = 1$ is a regular singular point.

(d) Determine whether the functions $y_1(x) = x^2$ and $y_2(x) = x|x|$ are linearly independent or not. Calculate its Wronskian.

(e) If $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$, show that $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2 = a^2(a^2+b^2)$.

(f) Prove that a necessary and sufficient condition for a vector function $\overrightarrow{a(t)}$ to have constant magnitude is $\vec{a} \cdot \frac{d\vec{a}}{dt} = 0$.

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) Define Lipschitz constant. Show that $f(x, y) = xy^2$ satisfies Lipschitz condition on the rectangle $|x| \le 1, |y| \le 1$. Find the Lipschitz constant. Does the function satisfy the Lipschitz condition on the strip $|x| \le 1, |y| < \infty$? Explain. 1+2+1+2

(b) Solve the following system of linear differential equations by using operator $D \equiv \frac{d}{dt}$ 6

$$\frac{dx}{dt}+\frac{dy}{dt}+2y=0$$
$$\frac{dx}{dt}-3x-2y=0$$

(c) Find the power series solution of the equation $(x^2+1)y''+xy'-xy=0$ in powers of x. 6

(d) Solve the differential equation $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$, it is given that $y=x+\frac{1}{x}$ is a solution of the differential equation. 6

(e) (i) A particle moves along the curve $x=2t^2$, $y=t^2-4t$, $z=3t-5$. Find the components of velocity and acceleration at time $t=1$, in the direction of $\hat{i}-3\hat{j}+2\hat{k}$. 4+2

(ii) Show that $\frac{d}{dt}\left(\vec{F}\times\frac{d\vec{F}}{dt}\right)=\vec{F}\times\frac{d^2\vec{F}}{dt^2}$, provided $\vec{F}$ and $\frac{d\vec{F}}{dt}$ are both differentiable.

(f) Find the directional derivative of $\varphi=2xy-z^2$ at $(2,-1,1)$ in the direction of $3\hat{i}+\hat{j}-\hat{k}$. In what direction is the directional derivative maximum? What is the value of the maximum? 3+1+2

GROUP-C

Answer any *two* questions from the following 12×2= 24

3. (a) Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x$. 6+6

(b) Solve by the method of undetermined coefficients $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=x^3e^{2x}+xe^{2x}$.

4. (a) Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+4y=\sec x\tan x$. 6+6

(b) Show that e^{2x} and e^{3x} are linearly independent solutions of $y''-5y'+6y=0$. Find the solution $y(x)$ with the conditions $y(0)=0$ and $y'(0)=1$.

5. (a) Find the singularities of the differential equation 3+3+6

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(1+\alpha+\beta)x\}\frac{dy}{dx}-\alpha\beta\gamma=0,$$

where α, β, γ are constants and determine the type of the singularities.

(b) Evaluate $\frac{1}{D^4+2D^3-3D^2}4\sin x$.

(c) Solve $(D^4+4)y=\sin 2x$.

6. (a) Show that $\vec{F} = 2xyz^2\hat{i} + (x^2z^2 + z\cos yz)\hat{j} + (2x^2yz + y\cos yz)\hat{k}$ is a conservative force field. 6+6

(b) If $\vec{F} = xy^2z\hat{i} + xy^3z\hat{j} - x^3y^2\hat{k}$ and $\vec{G} = x^3\hat{i} - x^2yz\hat{j} - x^2z^2\hat{k}$, calculate $\frac{\partial^2\vec{F}}{\partial y^2} \times \frac{\partial^2\vec{G}}{\partial x^2}$ at the point $(1, -1, 1)$.

MATHGE4

GROUP THEORY

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Prove that a group $(G, \circ)$ is abelian iff $(a \circ b)^{-1} = a^{-1} \circ b^{-1}$, $\forall a, b \in G$. 3

(b) Let G be a group of order pq, where p and q are distinct primes. Prove that every proper subgroup of G is cyclic. 3

(c) Let $G = (\mathbb{Z}_6, +)$, $H = \{\bar{0}, \bar{2}\}$. Check whether H is a normal subgroup of G or not? 3

(d) Let H be a subgroup of a group G and $[G:H] = 2$. Prove that H is normal in G. 3

(e) If G is a group such that $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$. Show that G must be abelian. 3

(f) Let $(G, *)$ be a group and $a, b \in G$. If $a^2 = e$ and $a * b^2 * a = b^3$, then prove that $b^5 = e$. 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) (i) Prove that there does not exist an onto homomorphism from the group $(\mathbb{Z}_6, +)$ to $(\mathbb{Z}_4, +)$. 3+3

(ii) Let $(G, \circ)$ be a group and a map $\varphi : G \to G$ is defined by $\varphi(x) = x^{-1}$, $x \in G$. Prove that φ is a homomorphism iff G is commutative.

(b) (i) Show that the union of two subgroups of a group G is not necessarily a subgroup of G. 2+4

(ii) Suppose that G be a group and H be a subgroup of G. Let $g \in G$ be fixed. Prove that the subset $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G.

(c) (i) If $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 3 & 4 & 2 & 5 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$ find $f \circ g$ and $g \circ f$. 2+4

(ii) Prove that every group of prime order is cyclic.

(d) Prove that the quotient group $Q/\mathbb{Z}$ is an infinite group, every element of which is of finite order. 6

(e) Let $GL(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc \neq 0 \right\}$. Find $C\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)$ and $Z(GL(2, \mathbb{R}))$. 3+3

(f) Suppose that $(G,\circ)$ is a group and $Z(G)=\{g \mid gx=xg, \forall x\in G\}$. Prove that $Z(G)$ is a normal subgroup of G. Also prove that if $G/Z(G)$ is cyclic then G is commutative. 3+3

GROUP-C

Answer any *two* questions from the following 12×2= 24

3. (a) Show that the set H forms a commutative group w.r.t matrix multiplication, when 4+2+3+3

$$H\left\{\begin{pmatrix} a & b \\ -b & a\end{pmatrix} : a,b\in\mathbb{R}\ ,\ a^2+b^2=1\right\}$$

(b) Examine the following system is a group or not? $(\mathbb{R},\circ)$ where $a\circ b=2(a+b)$, $a,b\in\mathbb{R}$.

(c) Prove that for a group $(G,\circ)$, $o(a)=b(a^{-1}), a\in G$.

(d) Suppose that the order of an element a in a group $(G,\circ)$ is 30. Find the order of a^{18}.

4. (a) Prove that two infinite cyclic group are isomorphic. 4+4+4

(b) Let $G=(\mathbb{R},+)$, $H=(\mathbb{Z},+)$ and $G'=(\{z\in\mathbb{C}:|z|=1\},\cdot)$ prove that $G/H\simeq G'$.

(c) Let $G=S_3$ and $G'=(\{-1,1\},\cdot)$ and $\varphi:G\to G'$ is defined by

$$\varphi(\alpha)=\begin{cases} 1, & \alpha \text{ is an even permutation in } S_3 \\ -1, & \alpha \text{ is an odd permutation in } S_3\end{cases}$$

Find $\ker(\varphi)$. Deduce that A_3 is a normal subgroup of S_3.

5. (a) Let G be a finite group generated by a. Prove that $O(G)=n$ iff $O(a)=n$. 6+2+4

(b) Write U_{10} and U_{12}. Show that U_{10} is a cyclic group but U_{12} is not cyclic.

6. (a) Let $A=\left\{\begin{pmatrix} a & a \\ a & a\end{pmatrix} : a\in\mathbb{R} \text{ and } a\neq 0\right\}$. Show that the set A forms a group under matrix multiplication. 6+2+4

(b) Give an example of a group which is abelian but not cyclic.

(c) Let α and β belongs to S_n. Prove that $\beta\times\beta^{-1}$ and α are both even or both odd.

MATHGE5

NUMERICAL METHODS

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) (i) Round off the following number to three decimal places: 20.1758 1

(ii) Find the number of significant figures in $X_A=1.8921$ given its relative error as 0.1×10^{-2}. 2

(b) What is the geometric representations of the trapezoidal rule for integrating $\int_a^b f(x)dx$. 3

(c) Find the function whose first difference is e^x tanning the step size $h=1$. 3

(d) If $u_0 = 1$, $u_1 = 11$, $u_2 = 21$, $u_3 = 28$, $u_4 = 29$ then find $\Delta^4 u_0$. 3

(e) If $h=1$ then find the value of $\Delta^3(1-x)(1-2x)(1-3x)$. 3

(f) Write down the order of convergence of 3

(i) Regula-Falsi method

(ii) Newton-Raphson method

(iii) Secant method

GROUP-B

Answer any *four* questions from the following 6×4 = 24

2. Use Euler's method to compute $y(0.04)$ from the differential equation $\frac{dy}{dx}+y=0$ with $y=1$, when $x=0$, taking $h=0.02$.

3. Solve by Gauss-Seidel iteration method the system

$$x_1 + x_2 + 4x_3 = 9$$
$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

upto three significant figure.

4. The third order differences of a function $f(x)$ are constant and $\int_{-1}^{1} f(x)dx = k\left[f(0)+f\left(\frac{1}{\sqrt{2}}\right)+f\left(-\frac{1}{\sqrt{2}}\right)\right]$ then find the value of *k*.

5. Find a real root of $x^x + x - 4 = 0$ by Newton-Raphson method, correct to six decimal places.

6. Find $f(0.23)$ from the following table using Newton's forward interpolation formula:

x	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

7. Evaluate $\int_1^4 \log_e \frac{(1+0.5x+x^2)}{0.5+x} dx$, by trapezoidal rule, correct upto six decimal, taking 13 ordinates points.

GROUP-C

Answer any *two* questions from the following 12×2= 24

8. (a) Compute $\int_{2}^{10} \frac{dx}{1+x}$ using Trapezoidal and Simpson's one third rule taking $h=1.0$ and compare the result with the exact value. 6

(b) Compute the root of the following by Regula-Falsi method 6

$2x-3\sin x-5=0$ correct upto three decimal places.

9. (a) A function $f(x)$ defined on $[0,1]$ such that $f(0)=0$, $f(1/2)=-1$, $f(1)=0$. Find the interpolating polynomial which approximate $f(x)$. 6

(b) Using Runge-Kutta method of order 2 to calculate $y(0.2)$ for the equation 6

$$\frac{dy}{dx}=x+y^2,\ y(0)=1$$

10.(a) Solve the following system of equations by Gauss-elimination method: 7

$$10x_1-7x_2+3x_3+5x_4=6\,,$$
$$-6x_1+8x_2-x_3-4x_4=5\,,$$
$$3x_1+x_2+4x_3+11x_4=2$$
$$5x_1-9x_2-2x_3+4x_4=7$$

(b) Evaluate the missing term in the following table 5

x	0	1	2	3	4	5
$f(x)$	0	?	8	15	?	35

11.(a) Solve the equations using Gauss-Jordan method: 4

$$3x_1+2x_2+3x_3=18$$
$$2x_1+x_2+x_3=10$$
$$x_1+4x_1+9x_3=16$$

(b) (i) What are 'partial and complete pivoting' in Gauss elimination method? 2

(ii) Find the number of multiplications and divisions for solving a system of n linear equations having n unknowns using Gauss-elimination method. 6

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